* Center of mass $(\bar{x}, \bar{y}, \bar{z})$

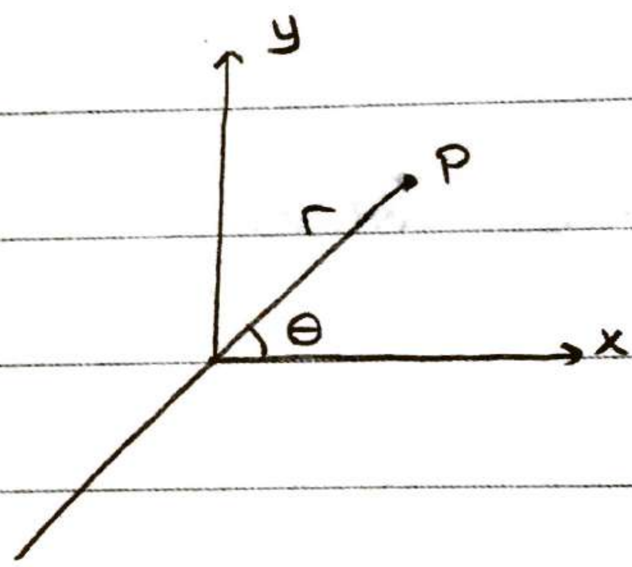
$$\bar{x} = \frac{M_{yz}}{M}$$

$$\bar{y} = \frac{M_{xz}}{M}$$

$$\bar{z} = \frac{M_{xy}}{M}$$

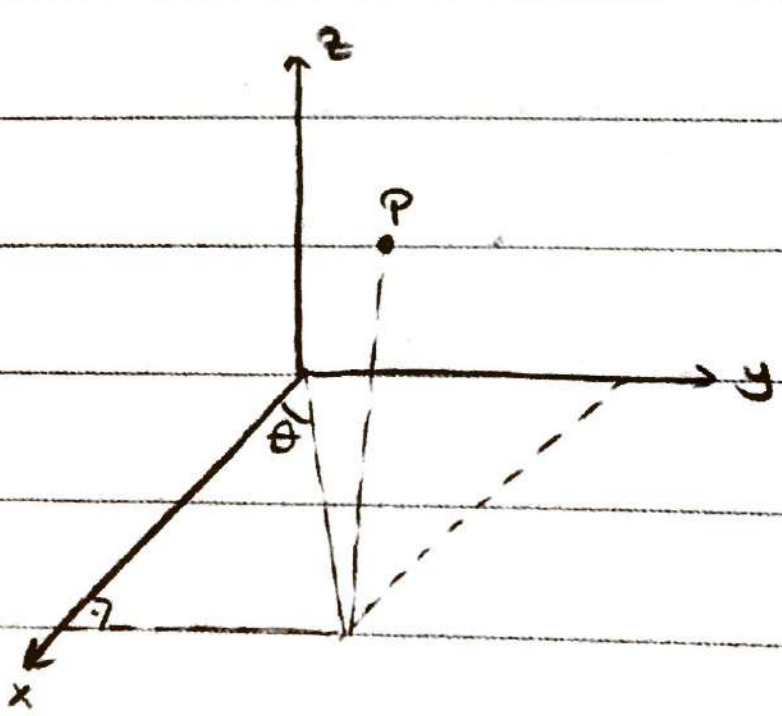
15.7 TRIPLE PRODUCTS IN CYLINDIRICAL AND SPHERICAL COORDINATES

* In Two Dimension (Polar coordinates)



$$\iint_R f(x,y)\,dA = \iint_{R'} f(r\cos\theta, r\sin\theta) = r\,dA'$$

* In Three Dimension (cylindirical coordinates)

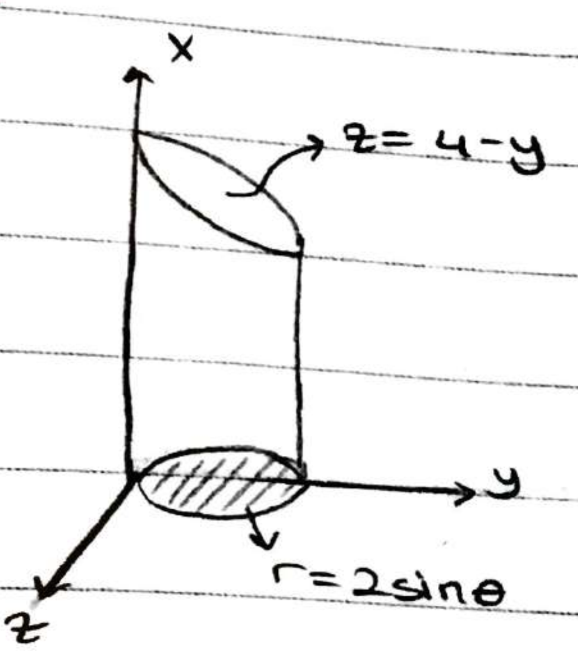


$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \begin{aligned} x^2 &= y^2 + z^2 \\ \tan\theta &= y/z \end{aligned}$$

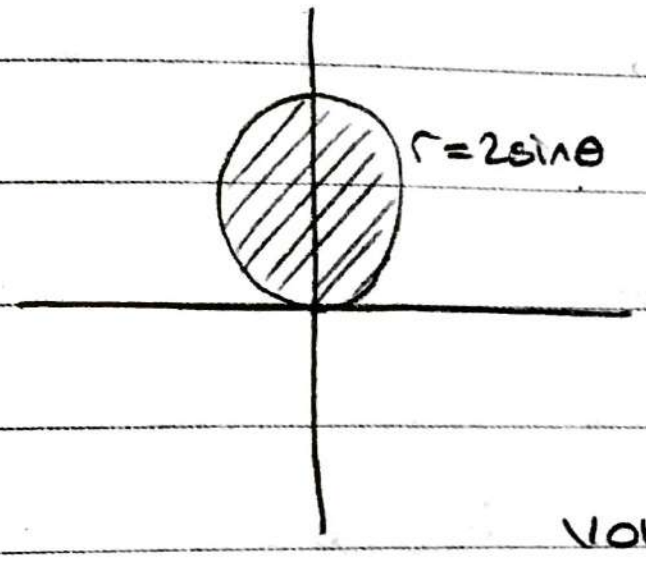
$$\iiint_R f(x,y,z)\,dV = \iiint_{R'} f(r\cos\theta, r\sin\theta, z) = r\,dV'$$

(book 15)

ex:



Find the volume?



$0 \leq \theta \leq \pi$

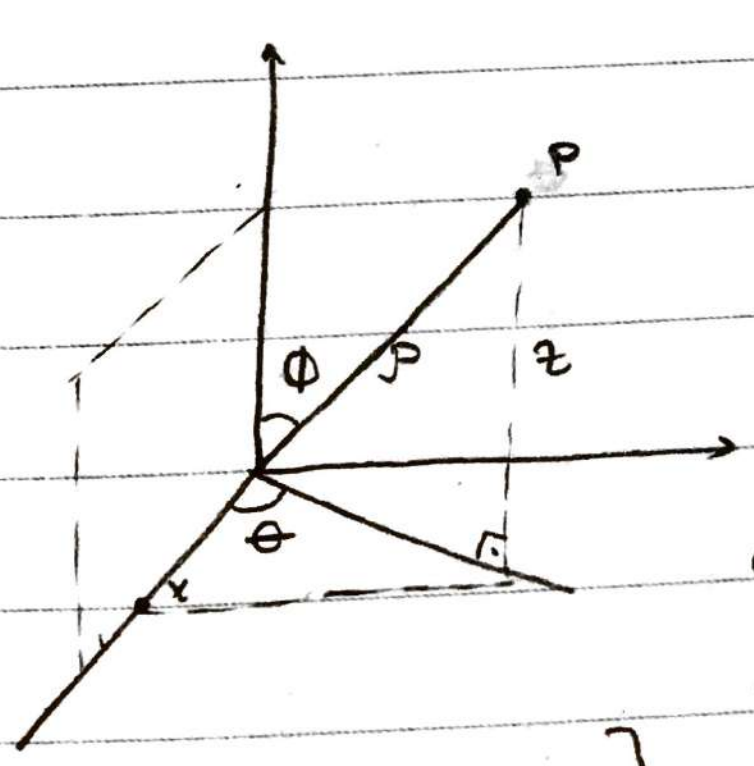
$0 \leq r \leq 2\sin\theta$

$0 \leq z \leq 4 - y = 4 - r\sin\theta$

$$\text{Volume} = \int_{\theta=0}^{\pi} \int_{r=0}^{2\sin\theta} \int_{z=0}^{(4-r\sin\theta)} 1 \, dz \cdot dr \cdot d\theta$$

$$\iiint_B f(x,y,z)\, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{(4-r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Spherical Coordinates



$z = \rho\cos\phi$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$0 \leq \phi \leq \pi$

$\phi = 0 \Rightarrow P$ is on the positive z-axis

$\phi = \pi \Rightarrow P$ is on the negative z-axis

$0 \leq \theta < 2\pi$

$\theta = 0 \Rightarrow f$ is on the xy plane

$\rho \rightarrow$ length of $\vec{OP}$

$\phi \rightarrow$ angle with positive z-axis

$\theta \rightarrow$ angle with positive x-axis

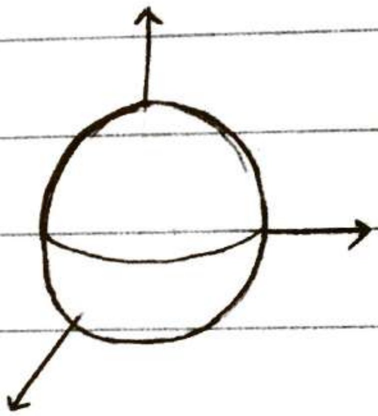
$$\rho^2 = x^2 + y^2 + z^2$$

ex: Find a spherical coordinate equation for the sphere $x^2+y^2+z^2=4$

$\hookrightarrow \rho = 2$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

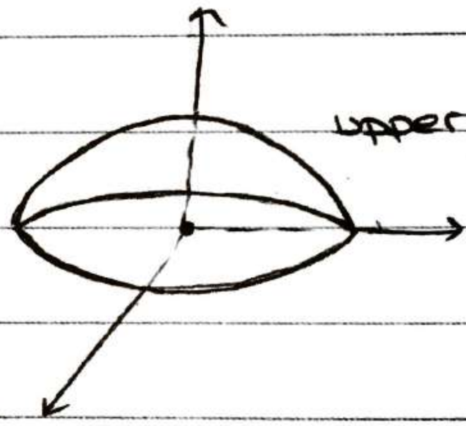


ex: $x^2+y^2+z^2=4$, $z \ge 0$

$\rho = 2$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/2$



upper hemisphere

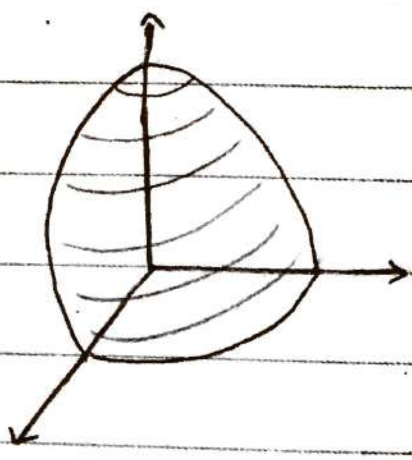
ex: $x^2+y^2+z^2=4$

$x \ge 0 \quad y \ge 0 \quad z \ge 0$

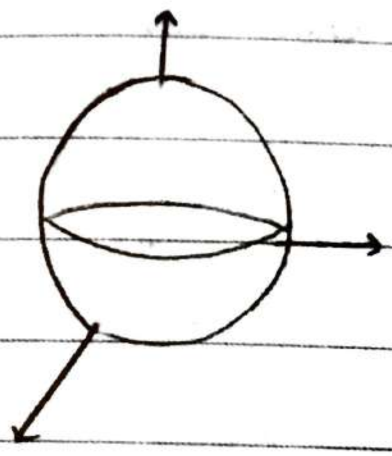
$\rho = 2$

$0 \le \theta \le \pi/2$

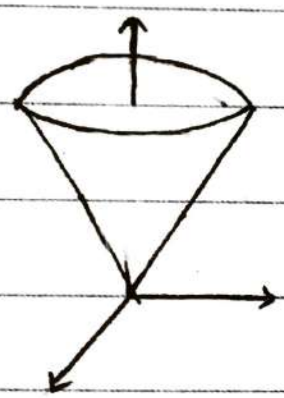
$0 \le \phi \le \pi/2$



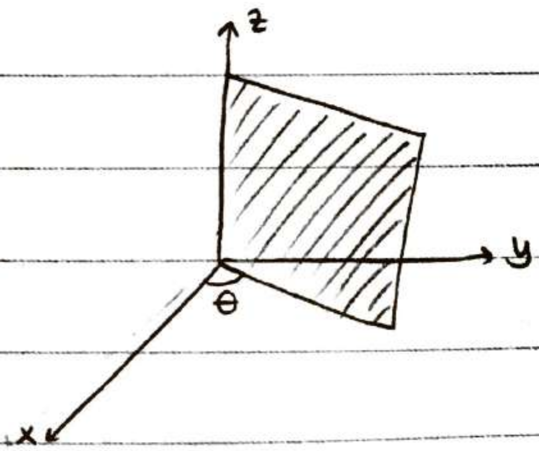
ρ = constant $\Rightarrow$ sphere



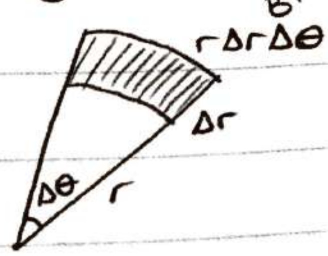
ϕ = constant $\Rightarrow$ cone



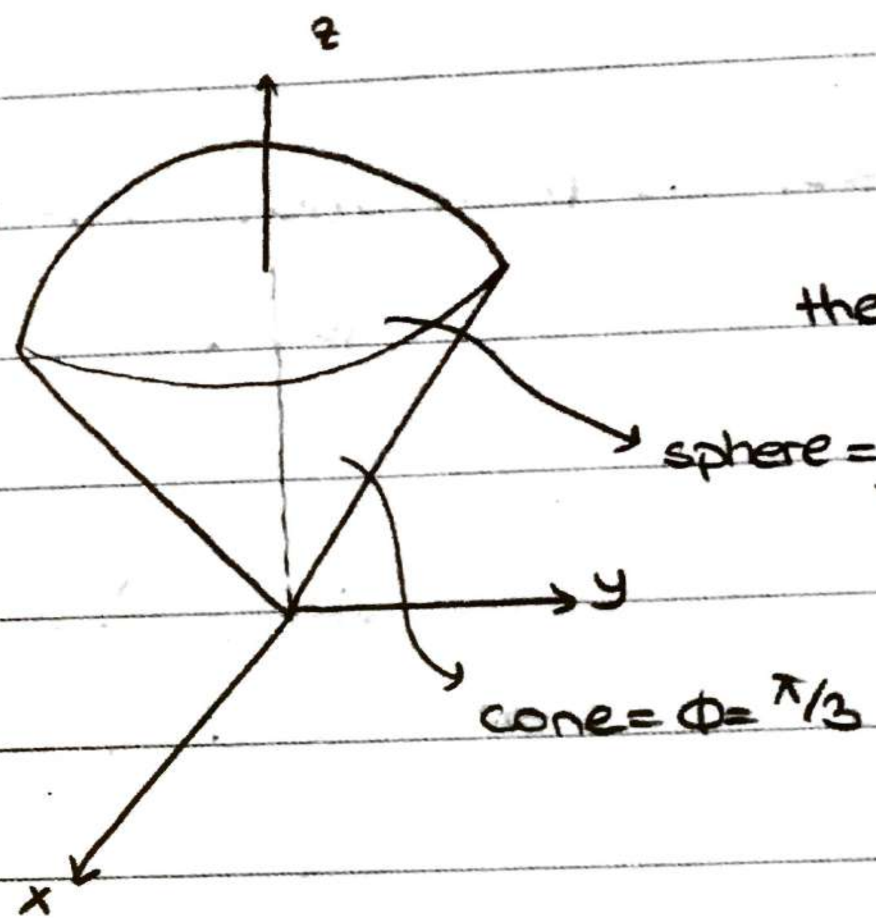
θ = constant



$$\iiint_B f(x,y,z)\,dV = \iiint_{B'} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\,dV'$$



ex:



Find the volume of the ice cream cone.

$0 \le \phi \le \pi/3$

$0 \le \theta \le 2\pi$

$0 \le \rho \le 1$

$$\text{Volume} = \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/3} 1 \cdot \rho^2 \sin\phi \cdot d\phi\, d\theta\, d\rho$$

$$= \int_{\rho=0}^{1} \rho^2 d\rho \int_{\theta=0}^{2\pi} d\theta \int_{\phi=0}^{\pi/3} \sin\phi\, d\phi$$

$$= 1/3 \cdot 2\pi \cdot \left(-\cos\frac{\pi}{3} + 1\right)$$

$$= \pi/3$$

ex: Find the volume inside the sphere with radius a

$0 \le \rho \le a$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

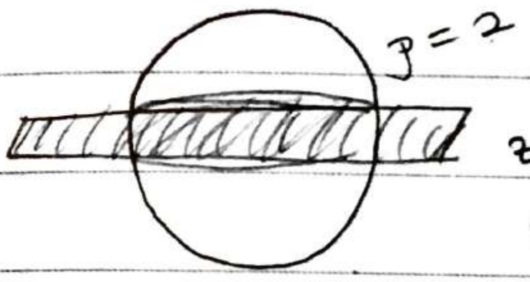
$$\text{Volume} = \int_{\rho=0}^{a} \rho^2 d\rho \int_{\theta=0}^{2\pi} d\theta \int_{\phi=0}^{\pi} \sin\phi\, d\phi$$

$$= \int_{\rho=0}^{a}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} 1 \cdot \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$$

$$= \frac{a^3}{3} \cdot 2\pi\,(-\cos\pi + 1) \cdot \frac{4\pi a^2}{3}$$

(book 51)

ex: Find the volume of the solid region cut from the solid sphere $\rho \leq 2$ by the plane $z = 1$



$\rho = 1 \cos \Phi$

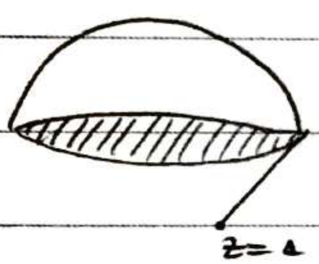
$0 \leq \Phi \leq \pi/3$

$0 \leq \theta \leq 2\pi$

$\frac{1}{\cos\Phi} \leq \rho \leq 2$

$z = \rho\cos\Phi = 1$

$\rho = \frac{1}{\cos\Phi}$



$\cos\Phi = \frac{2}{\rho} = \frac{1}{2} \Rightarrow \pi/3$

$$\text{Volume} = \int_{\theta=0}^{2\pi} \int_{\Phi=0}^{\pi/3} \int_{\rho=1/\cos\Phi}^{2} \rho^2 \sin\Phi \, d\rho \, d\Phi \, d\theta$$

15.8 → SUBSTITUTIONS IN MULTIPLE INTEGRALS

1 Variable Case (calculus 1)

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_{u=a}^{b} f(g(u)) \cdot g'(u)\,du$$

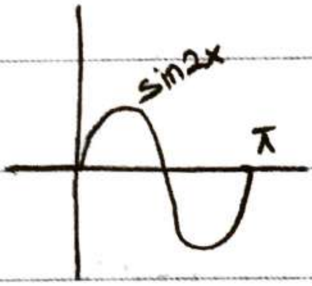
$x \Rightarrow g(u)$

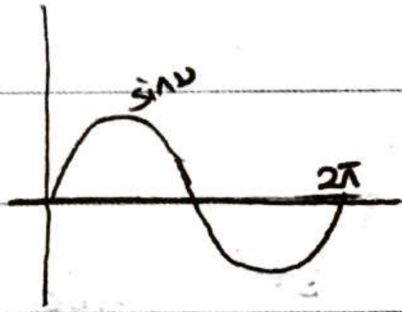
$dx \Rightarrow g'(u)du$

$x = g(a) \Rightarrow g(u) = g(a) \Rightarrow u = a$

ex: $\displaystyle\int_{x=0}^{\pi} \sin(2x)\,dx = \int_{u=0}^{2\pi} (\sin u)\frac{1}{2}\,du$

$x \to u/2$





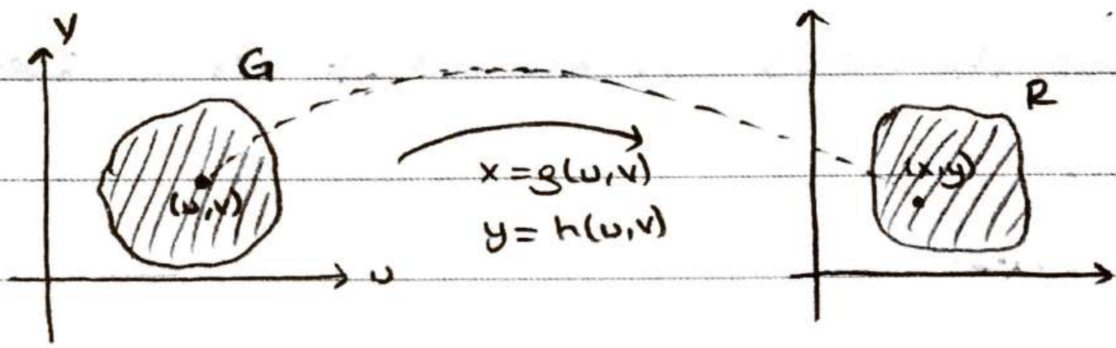
2 variable case

$x = g(u,v)$ transformation

$y = h(u,v)$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$



$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

ex: Find the Jacobian for polar coordinates transformation

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial x}{\partial \theta} \end{vmatrix} = \cos\theta . r\cos\theta - (-r\sin\theta)\sin\theta$$

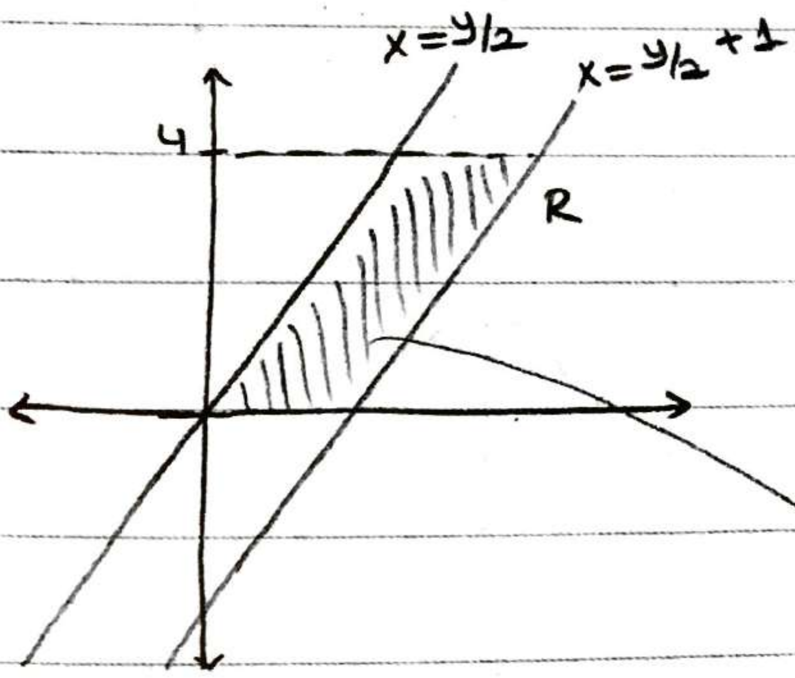
$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

ex: Evaluate

$$\int_0^4 \int_{x=y/2}^{x=y/2+1} \frac{2x-y}{2}\, dx\, dy$$

by applying $u = \dfrac{2x-y}{2} \rightarrow x = u+v$ $\quad v = \dfrac{y}{2} \rightarrow y = 2v$

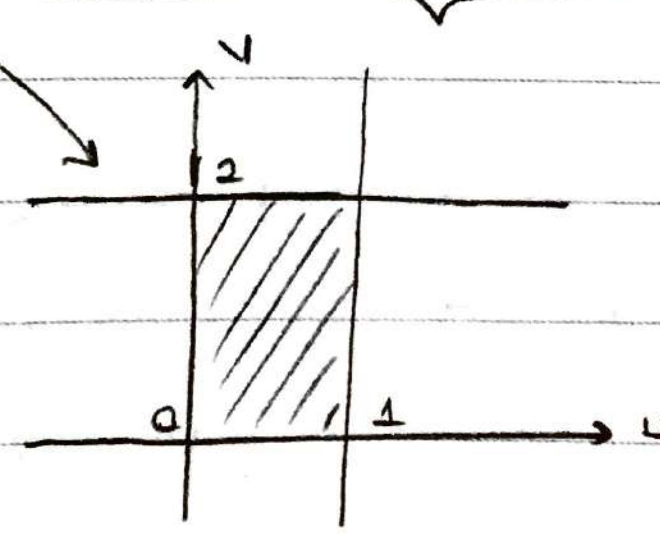


$\circledast$ $x = \dfrac{y}{2} \Rightarrow u = 0$

$\circledast$ $x = \dfrac{y}{2} + 1 \Rightarrow u = 1$

$\circledast$ $y = 0 \Rightarrow v = 0$

$y = 4 \Rightarrow v = 2$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$$\iint_R \frac{2x-y}{2}\, dx\, dy = \iint_G u . 2\, du\, dv \qquad 0 \leq u \leq 1$$

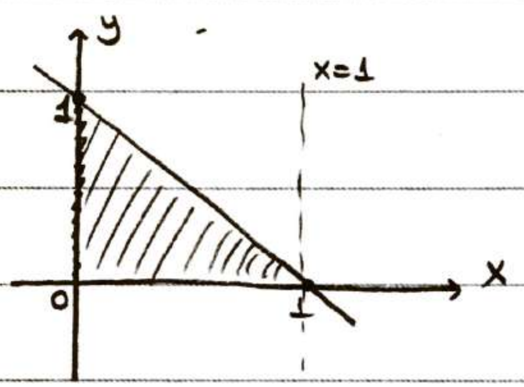
$$= \int_0^1 \int_0^2 2u\, dv\, du \qquad 0 \leq v \leq 2$$

$$= \int_0^1 2u . 2\, du = 2$$

$$\text{Area}(R) = \iint_R 1\,dx\,dy = \iint_G 1\,J\,du\,dv$$

ex: $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

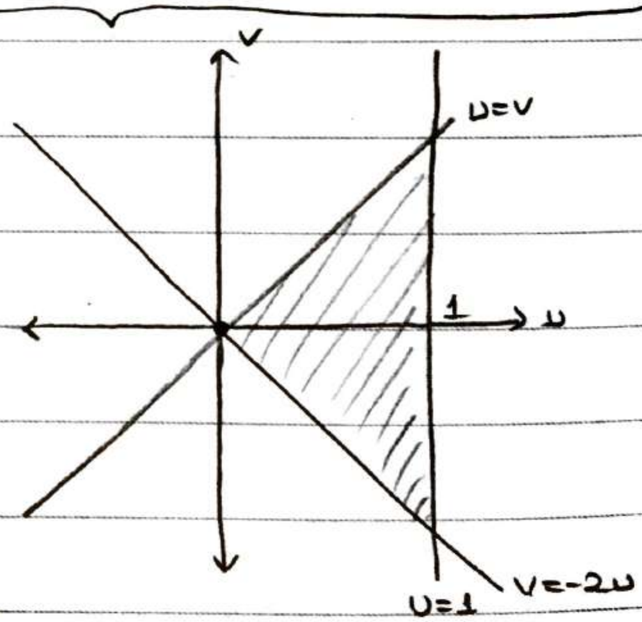
$^2/$ $\left.\begin{aligned} x+y &= u \\ y-2x &= v \end{aligned}\right\} \quad \left.\begin{aligned} 2x+2y &= 2u \\ y-2x &= v \end{aligned}\right\} \quad y = \dfrac{2u+v}{3} \qquad x = \dfrac{u-v}{3}$



$x=0 \Rightarrow \left.\begin{aligned} u&=y \\ v&=y \end{aligned}\right\} \; u=v$

$y=1-x \Rightarrow u=1$

$y=0 \Rightarrow \left.\begin{aligned} u&=x \\ v&=-2x \end{aligned}\right\} \; v=-2u$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \underbrace{\frac{1}{3}}_{\frac{\partial x}{\partial u}} \cdot \underbrace{\frac{1}{3}}_{\frac{\partial y}{\partial v}} - \left(\underbrace{-\frac{1}{3}}_{\frac{\partial x}{\partial v}}\right) \cdot \underbrace{\frac{2}{3}}_{\frac{\partial y}{\partial u}}$$
$$= 1/3$$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

$$= \int_{u=0}^{1} \int_{v=-2u}^{u} \sqrt{u} \cdot v^2 \cdot \overset{J}{\frac{1}{3}}\,dv\,du$$

$0 \leq u \leq 1$

$-2u \leq v \leq u$

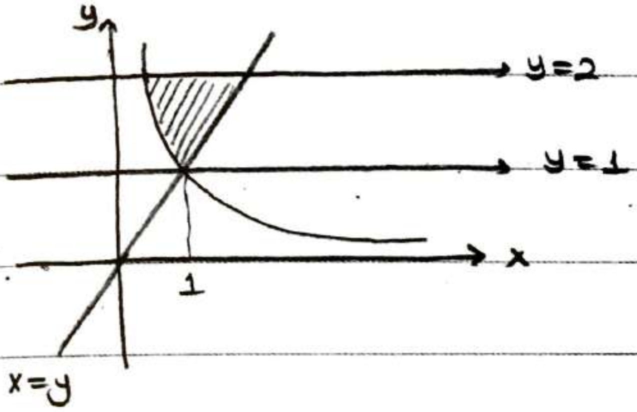
$$= 2/9$$

ex: $\displaystyle\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dx\,dy$

$y>0,\ x>0,\ u>0,\ v>0$

$\left.\begin{array}{l}\sqrt{xy}=u\\ \sqrt{\frac{y}{x}}=v\end{array}\right\}\ y^2=u^2v^2 \Rightarrow y=uv$

$x=u/v$

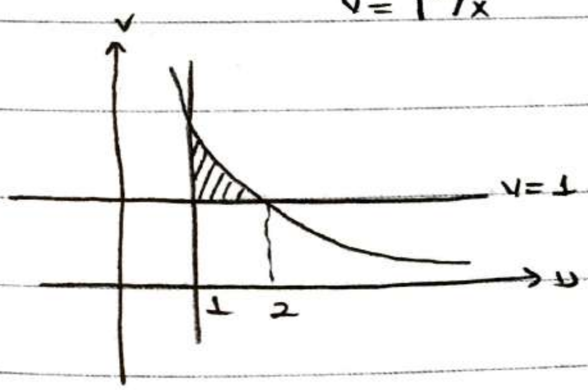


⊛ $x=1/y \Rightarrow u=\sqrt{1}=1$

⊛ $y=x \Rightarrow y/x=1 \Rightarrow v=\sqrt{1}=1$

⊛ $y=2 \Rightarrow u=\sqrt{2x} \Rightarrow \left.\sqrt{x}=\frac{u}{\sqrt{2}}=\frac{\sqrt{2}}{v}\right\}\ u.v=2$

$v=\sqrt{2/x}$



$1\le u\le 2$

$1\le v\le 2/u$

$J=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

$=\frac{1}{v}.u-\left(\frac{-u}{v^2}\right)v=\frac{2u}{v}$

$\displaystyle\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dx\,dy=\int_{u=1}^{2}\int_{v=1}^{2/u} v\,e^{u}\,\frac{2u}{v}\,dv\,du$

$=2e(e-2)$